Math 565: Functional Analysis
Lecture 17

Dual of a top. v.s. X and weak* topology.

For a top. v.s. X, let X^* denote the vector space of continuous functionals on X.

Remark. As mentioned last time, when the top. on X is generated by seminorms, X^* is a rich space by Hahn-Banach.

To equip X^* with a reasonable topology, recall that $X^* \subseteq \mathbb{C}^X$ and equip X^* with the subspace topology of the product/pointwise convergence top on $\mathbb{C}^X$. We call this the **weak*** topology and write $f_n \to_* f$ for weak* convergence.

Obs. (a) $f_n \to_* f \;:\Leftrightarrow\; f_n(x) \to f(x)$ for all $x \in X$.

(b) Weak* top is generated by sets $[x \mapsto B] := \{f \in X^* : f(x) \in B\}$ where $x \in X$, $B \subseteq \mathbb{C}$ is an open ball.

(c) A basis for weak* top is finite intersections of the sets $[x \mapsto B]$ i.e.
$$[x_1 \mapsto B_1, x_2 \mapsto B_2, \dots, x_n \mapsto B_n] := \bigcap_{i=1}^{n} [x_i \mapsto B_i].$$

(d) Weak* top is Hausdorff since $\mathbb{C}^X$ is.

Obs. When X is a normed vector space, X^* is also equipped with the norm topology, which is stronger (has more open sets) than weak* topology because weak* is generated by functions $f \mapsto f(x) : X^* \to \mathbb{C}$ and these functions are continuous in the norm topology, in fact, they are Lipschitz constant $\|x\|$:
$$|f(x) - g(x)| = |(f-g)(x)| \leq \|x\| \cdot \|f-g\|.$$

Example. Let $X := \ell^1$ so $(\ell^1)^*$ is isometrically isom. to ℓ^∞, i.e. the norm on $(\ell^1)^*$ is the same as the ℓ^∞-norm. Let $(e_n) \subseteq \ell^\infty$, where $e_n := \mathbb{1}_{\{n\}}$. Note that $\|e_n - e_m\| = 1$ when $n \neq m$, so this sequence does not converge in norm. However, $e_n \to_* 0$ because for each $x \in \ell^1$, $x(n) \to 0$ and e_n evaluated at x is $\sum_{i \in \mathbb{N}} e_n(i)\, x(i) = x(n)$.

Although weak* on X^* makes sense for any top. v.s. X, it has most structure when X is a normed vector space. Thus, from now on, we focus on normed vector spaces. We will show that in ∞-dim spaces, weak* is strictly weaker than the norm top.

Clearly, $X^* \subseteq \mathrm{Lin}(X, \mathbb{C}) \subseteq \mathbb{C}^X$, where $\mathrm{Lin}(X, \mathbb{C}) := \{ f \in \mathbb{C}^X : f \text{ is linear}\}$. The set $\mathrm{Lin}(X, \mathbb{C})$ is closed because not being linear is witnessed by two/three coordinates and such sets are open in the product top. Let $\overline{B}_1^X$ denote the closed unit ball in X.

Prop. The restriction map $f \mapsto f_{\overline{B}_1^X} : \mathrm{Lin}(X, \mathbb{C}) \to \mathrm{Lin}(\overline{B}_1^X, \mathbb{C}) \subseteq \mathbb{C}^{\overline{B}_1^X}$ is a homeomorphism.

Proof. Firstly, it is a bijection because $f(x) = \|x\| f(\frac{1}{\|x\|} x)$ for all $x \notin \overline{B}_1^X$. It is also a homeomorphism because for any net $(f_i)_{i \in I} \subseteq \mathrm{Lin}(X, \mathbb{C})$ and $f \in \mathrm{Lin}(X, \mathbb{C})$,

$$f_i \to f \text{ pointwise} \iff f_i|_{\overline{B}_1^X} \to f|_{\overline{B}_1^X} \text{ pointwise.}$$

This shows that both the restriction map and its inverse are continuous. $\square$

For $r \geq 0$, let $\overline{B}_r^* := \{ f \in X^* : \|f\| \leq r\}$ denote the radius r closed unit ball in X^* (wrt the norm).

Cor. For $r \geq 0$, $\overline{B}_r^*$ is homeomorphic to $\mathrm{Lin}(\overline{B}_1^X, \overline{B}_r^{\mathbb{C}})$ via the restriction map $f \mapsto f|_{\overline{B}_1^X}$. In particular, X^* is homeomorphic to $\bigcup_{n \in \mathbb{N}} \mathrm{Lin}(\overline{B}_1^X, \overline{B}_n^{\mathbb{C}})$.

Proof. If $f \in \overline{B}_r^*$ then $f(\overline{B}_1^X) \subseteq \overline{B}_r^{\mathbb{C}}$ because $|f(x)| \leq \|f\| \|x\| \leq r$ for all $x \in \overline{B}_1^X$. And if $f \in X^*$, then $f|_{\overline{B}_1^X} \in \mathrm{Lin}(\overline{B}_1^X, \overline{B}_n^{\mathbb{C}})$ for any $n \geq \|f\|$. $\square$

Banach-Alaoglu Obs. $\overline{B}_1^*$ is compact in the weak* top.

Proof. $\overline{B}_1^*$ in the weak* top is homeom. to $\mathrm{Lin}(\overline{B}_1^X, \overline{B}_1^{\mathbb{C}})$ which is a closed subset of $(\overline{B}_1^{\mathbb{C}})^{\overline{B}_1^*}$ and the latter space is compact by Tychonoff's theorem, so $\mathrm{Lin}(\overline{B}_1^X, \overline{B}_1^{\mathbb{C}})$ is compact as well because closed subsets of compact spaces are compact. $\square$

Prop. If X is separable, then the weak* top on $\overline{B}_1^*$ is metrizable. Thus, $\overline{B}_1^*$ is sequentially compact in the weak* top (i.e. every sequence has a convergent subsequence).

Proof. Let $\mathcal{D}$ be a ctbl dense set in X, so $\mathcal{D}_1 := \mathcal{D} \cap B_1^X$ is dense in $\overline{B}_1^X$. By the continuity of each $f \in X^*$, f is determined by its values on $\mathcal{D}$, and $f|_{\overline{B}_1^X}$ is determined by $f|_{\mathcal{D}_1}$. Thus, one can show that $\overline{B}_1^* \cong \mathrm{Lin}(\overline{B}_1^X, \overline{B}_1^{\mathbb{C}}) \cong \mathrm{Lin}(\mathcal{D}_1, \overline{B}_1^{\mathbb{C}})$, which is a closed subset of $(\overline{B}_1^{\mathbb{C}})^{\mathcal{D}_1}$. But ctbl products of metric spaces are metrizable, so $\overline{B}_1^*$ is metrizable. The details will appear in HW. $\square$

Examples. Recall that for an lcH top. space X, $C_0(X)^* \approx RM(X) :=$ the space of all complex Radon measures on X. For $(\mu_n) \subseteq RM(X)$ and $\mu \in RM$,

$$\mu_n \to_* \mu \quad :\Leftrightarrow \quad \int f \, d\mu_n \to \int f \, d\mu \text{ for all } f \in C_0(X).$$

This is called vague convergence of measures.

(a) Let $(x_n) \subseteq X$ so $\delta_{x_n} \in RM(X)$. Then $\mu_n := \frac{1}{n} \sum_{i<n} \delta_{x_i}$ are still probability measures so $(\mu_n) \subseteq \overline{B}_1^*$ hence they have a convergent subnet, and also a convergent subsequence if $C_0(X)$ is separable (e.g. when X is 2^{nd} ctbl). Let μ be some limit.

For example, if $X := \mathbb{R}$ and $\mu_n := \sum_{k=0}^{2n^2-1} \delta_{-n+k\frac{1}{n}}$, then $\mu_n \to_*$ Lebesgue measure.

(b) Let $V := \mathbb{N}$ and $E := \mathbb{N}^2$. Put $X := 2^E$ (so X is the Cantor space, hence compact and 2^{nd} ctbl). Then each point $G \in 2^E$ is just a directed graph on V. Probability measures on 2^E are called random graphs on V. Let μ_n be a prob. meas. supported the (finite) set of graph on $n := \{0, 1, \dots, n-1\}$. Then (μ_n) has a convergent subsequence, whose limit a random graph which is infinite a.s.